Lagrangian Mechanics on the standard Cliffordian Kähler Manifolds

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Abstract

This study presents standard Cliffordian Kähler analogue of Lagrangian mechanics. Also, the some geometric and physical results related to the standard Cliffordian Kähler dynamical systems are given.

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1 Introduction

Modern differential geometry explains explicitly the dynamics of Lagrangians. So, we say that if M is an m-dimensional configuration manifold and $L : TM \to \mathbf{R}$ is a regular Lagrangian function, then there is a unique vector field ξ on TM such that dynamics equations is given by

$$i_\xi \Phi_L = dE_L \tag{1}$$

where Φ_L indicates the symplectic form. The triple (TM, Φ_L, ξ) is called *Lagrangian system* on the tangent bundle TM .

In literature, there are a lot of studies about Lagrangian mechanics, formalisms, systems and equations [1, 2] and there in. There are real, complex, paracomplex and other analogues. It is possible to produce different analogous in different spaces. Finding new dynamics equations is both a new expansion and contribution to science to explain physical events.

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex numbers. Hamilton's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra. It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics [3] . Cliffordian manifold is a quaternion manifold. The above properties yield also for Cliffordian manifold.

In this paper, Euler-Lagrange equations related to Lagrangian systems on Cliffordian Kähler manifold have been obtained.

2 Preliminaries

Throughout this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M)$, $\chi(M)$ and $\Lambda^1(M)$ denote the set of functions on M, the set of vector fields on M and the set of 1-forms on M, respectively.

2.1 Cliffordian Kähler Manifolds

Here, we recalled the main concepts and structures given in [4, 5] . Let M be a real smooth manifold of dimension m. Suppose that there is a 6-dimensional vector bundle V consisting of $F_i(i = 1, 2, ..., 6)$ tensors of type (1,1) over M. Such a local basis $\{F_1, F_2, ..., F_6\}$ is called a canonical local basis of the bundle V in a neighborhood U of M. Then V is called an almost Cliffordian structure in M. The pair (M, V) is named an almost Cliffordian manifold with V. Hence, an almost Cliffordian manifold M is of dimension $m = 8n$. If there exists on (M, V) a global basis $\{F_1, F_2, ..., F_6\}$, then (M, V) is said to be an almost Cliffordian manifold; the basis $\{F_1, F_2, ..., F_6\}$ is called a global basis for V.

An almost Cliffordian connection on the almost Cliffordian manifold (M, V) is a linear connection ∇ on M which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local-global) of the bundle V, then $\nabla_X\Phi$ is also a cross-section (local-global, respectively) of V, X being an arbitrary vector field of M.

If for any canonical basis $\{J_1, J_2, ..., J_6\}$ of V in a coordinate neighborhood U, the identities

$$g(J_iX, J_iY) = g(X, Y),\ \forall X, Y \in \chi(M), \quad i = 1, 2, ..., 6, \tag{3}$$

hold, the triple (M, g, V) is named an almost Cliffordian Hermitian manifold or metric Cliffordian manifold denoting by V an almost Cliffordian structure V and by g a Riemannian metric and by (g, V) an almost Cliffordian metric structure.

Since each $J_i (i = 1, 2, ..., 6)$ is almost Hermitian structure with respect to g, setting

$$\Phi_i(X, Y) = g(J_iX, Y), \quad i = 1, 2, ..., 6, \tag{4}$$

for any vector fields X and Y, we see that Φ_i are 6 local 2-forms.

If the Levi-Civita connection $\nabla = \nabla^g$ on (M, g, V) preserves the vector bundle V by parallel transport, then (M, g, V) is called a Cliffordian Kähler manifold, and an almost Cliffordian structure Φ_i of M is called a Cliffordian Kähler structure. A Clifford Kähler manifold is Riemannian manifold (M^{8n}, g). For example, we say that $\mathbf{R}^{8n}$ is the simplest example of Clifford Kähler manifold. Suppose that let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\}$, $i = \overline{1, n}$ be a real coordinate system on $\mathbf{R}^{8n}$. Then we define by $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}, \frac{\partial}{\partial x_{4n+i}}, \frac{\partial}{\partial x_{5n+i}}, \frac{\partial}{\partial x_{6n+i}}, \frac{\partial}{\partial x_{7n+i}} \right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}, dx_{4n+i}, dx_{5n+i}, dx_{6n+i}, dx_{7n+i}\}$ be natural bases over $\mathbf{R}$ of the tangent space $T(\mathbf{R}^{8n})$ and the cotangent space $T^*(\mathbf{R}^{8n})$ of $\mathbf{R}^{8n}$, respectively. By structures

J_1, J_2, J_3, the following expressions are obtained

$$\begin{aligned}
&J_1(\tfrac{\partial}{\partial x_i}) = \tfrac{\partial}{\partial x_{n+i}},\ J_1(\tfrac{\partial}{\partial x_{n+i}}) = -\tfrac{\partial}{\partial x_i},\ J_1(\tfrac{\partial}{\partial x_{2n+i}}) = \tfrac{\partial}{\partial x_{4n+i}},\ J_1(\tfrac{\partial}{\partial x_{3n+i}}) = \tfrac{\partial}{\partial x_{5n+i}},\\
&J_1(\tfrac{\partial}{\partial x_{4n+i}}) = -\tfrac{\partial}{\partial x_{2n+i}},\ J_1(\tfrac{\partial}{\partial x_{5n+i}}) = -\tfrac{\partial}{\partial x_{3n+i}},\ J_1(\tfrac{\partial}{\partial x_{6n+i}}) = \tfrac{\partial}{\partial x_{7n+i}},\ J_1(\tfrac{\partial}{\partial x_{7n+i}}) = -\tfrac{\partial}{\partial x_{6n+i}},\\
&J_2(\tfrac{\partial}{\partial x_i}) = \tfrac{\partial}{\partial x_{2n+i}},\ J_2(\tfrac{\partial}{\partial x_{n+i}}) = -\tfrac{\partial}{\partial x_{4n+i}},\ J_2(\tfrac{\partial}{\partial x_{2n+i}}) = -\tfrac{\partial}{\partial x_i},\ J_2(\tfrac{\partial}{\partial x_{3n+i}}) = \tfrac{\partial}{\partial x_{6n+i}},\\
&J_2(\tfrac{\partial}{\partial x_{4n+i}}) = \tfrac{\partial}{\partial x_{n+i}},\ J_2(\tfrac{\partial}{\partial x_{5n+i}}) = -\tfrac{\partial}{\partial x_{7n+i}},\ J_2(\tfrac{\partial}{\partial x_{6n+i}}) = -\tfrac{\partial}{\partial x_{3n+i}},\ J_2(\tfrac{\partial}{\partial x_{7n+i}}) = \tfrac{\partial}{\partial x_{5n+i}},\\
&J_3(\tfrac{\partial}{\partial x_i}) = \tfrac{\partial}{\partial x_{3n+i}},\ J_3(\tfrac{\partial}{\partial x_{n+i}}) = -\tfrac{\partial}{\partial x_{5n+i}},\ J_3(\tfrac{\partial}{\partial x_{2n+i}}) = -\tfrac{\partial}{\partial x_{6n+i}},\ J_3(\tfrac{\partial}{\partial x_{3n+i}}) = -\tfrac{\partial}{\partial x_i},\\
&J_3(\tfrac{\partial}{\partial x_{4n+i}}) = \tfrac{\partial}{\partial x_{7n+i}},\ J_3(\tfrac{\partial}{\partial x_{5n+i}}) = \tfrac{\partial}{\partial x_{n+i}},\ J_3(\tfrac{\partial}{\partial x_{6n+i}}) = \tfrac{\partial}{\partial x_{2n+i}},\ J_3(\tfrac{\partial}{\partial x_{7n+i}}) = -\tfrac{\partial}{\partial x_{4n+i}}.
\end{aligned} \tag{5}$$

3 Lagrangian Mechanics

In this section, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{J_1, J_2, J_3\}$ of V on the standard Cliffordian Kähler manifold $(\mathbf{R}^{8n}, V)$.

Firstly, let J_1 take a local basis component on the standard Cliffordian Kähler manifold $(\mathbf{R}^{8n}, V)$, and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\}$, $i = \overline{1,n}$ be its coordinate functions. Let semispray be the vector field ξ determined by

$$\begin{aligned}
\xi = {} & X^i\tfrac{\partial}{\partial x_i} + X^{n+i}\tfrac{\partial}{\partial x_{n+i}} + X^{2n+i}\tfrac{\partial}{\partial x_{2n+i}} + X^{3n+i}\tfrac{\partial}{\partial x_{3n+i}}\\
& + X^{4n+i}\tfrac{\partial}{\partial x_{4n+i}} + X^{5n+i}\tfrac{\partial}{\partial x_{5n+i}} + X^{6n+i}\tfrac{\partial}{\partial x_{6n+i}} + X^{7n+i}\tfrac{\partial}{\partial x_{7n+i}},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
X^i &= \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i},\\
X^{4n+i} &= \dot{x}_{4n+i}, X^{5n+i} = \dot{x}_{5n+i}, X^{6n+i} = \dot{x}_{6n+i}, X^{7n+i} = \dot{x}_{7n+i}
\end{aligned}$$

and the dot indicates the derivative with respect to time t. The vector fields defined by

$$\begin{aligned}
V_{J_1} = J_1(\xi) = {} & X^i\tfrac{\partial}{\partial x_{n+i}} - X^{n+i}\tfrac{\partial}{\partial x_i} + X^{2n+i}\tfrac{\partial}{\partial x_{4n+i}} + X^{3n+i}\tfrac{\partial}{\partial x_{5n+i}}\\
& - X^{4n+i}\tfrac{\partial}{\partial x_{2n+i}} - X^{5n+i}\tfrac{\partial}{\partial x_{3n+i}} + X^{6n+i}\tfrac{\partial}{\partial x_{7n+i}} - X^{7n+i}\tfrac{\partial}{\partial x_{6n+i}},
\end{aligned} \tag{7}$$

is called *Liouville vector field* on the standard Cliffordian Kähler manifold $(\mathbf{R}^{8n}, V)$. The maps given by $T, P : \mathbf{R}^{8n} \to \mathbf{R}$ such that

$$T = \frac{1}{2} m_i(\dot{x}_i^2 + \dot{x}_{n+i}^2 + x_{2n+i}^2 + \dot{x}_{3n+i}^2 + \dot{x_{4n+i}}^2 + \dot{x}_{5n+i}^2 + x_{6n+i}^2 + \dot{x}_{7n+i}^2), \quad P = m_i g h$$

are called *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on the standard Cliffordian Kähler manifold $(\mathbf{R}^{8n}, V)$, respectively. Then $L : \mathbf{R}^{8n} \to R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function, ii)* the function given by $E_L^{J_1} = V_{J_1}(L) - L$, is *energy function.*

The operator i_{J_1} induced by J_1 and given by

$$i_{J_1}\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., J_1 X_i, ..., X_r), \tag{8}$$

is said to be *vertical derivation,* where $\omega \in \wedge^r \mathbf{R}^{8n}$, $X_i \in \chi(\mathbf{R}^{8n})$. The *vertical differentiation* d_{J_1} is defined by

$$d_{J_1} = [i_{J_1}, d] = i_{J_1} d - d i_{J_1} \tag{9}$$

where d is the usual exterior derivation. For J_1 , the closed Cliffordian Kähler form is the closed 2-form given by $\Phi_L^{J_1} = -dd_{J_1} L$ such that

$$\begin{aligned} d_{J_1} \;=\; & \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{4n+i}} dx_{2n+i} + \frac{\partial}{\partial x_{5n+i}} dx_{3n+i} \\ & - \frac{\partial}{\partial x_{2n+i}} dx_{4n+i} - \frac{\partial}{\partial x_{3n+i}} dx_{5n+i} + \frac{\partial}{\partial x_{7n+i}} dx_{6n+i} - \frac{\partial}{\partial x_{6n+i}} dx_{7n+i} \end{aligned}$$

defined by operator

$$d_{J_1} : \mathcal{F}(\mathbf{R}^{8n}) \to \wedge^1 \mathbf{R}^{8n}. \tag{10}$$

Then

$$\Phi_L^{J_1} = -\frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j \wedge dx_{2n+i}$$

$$-\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{5n+i}$$

$$-\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j \wedge dx_{6n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_i$$

$$+\frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} \wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{5n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} \wedge dx_{6n+i}$$

$$+\frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{n+i}$$

$$-\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{2n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{2n+j} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{4n+i}$$

$$+\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{5n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} dx_{2n+j} \wedge dx_{6n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} dx_{2n+j} \wedge dx_{7n+i}$$

$$-\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{3n+j} \wedge dx_{2n+i}$$

$$-\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{3n+j} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{5n+i}$$

$$-\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} dx_{3n+j} \wedge dx_{6n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} dx_{3n+j} \wedge dx_{7n+i} - \frac{\partial L}{\partial x_{4n+j} \partial x_{n+i}} dx_{4n+j} \wedge dx_i$$

$$+\frac{\partial L}{\partial x_{4n+j} \partial x_i} dx_{4n+j} \wedge dx_{n+i} - \frac{\partial L}{\partial x_{4n+j} \partial x_{4n+i}} dx_{4n+j} \wedge dx_{2n+i} - \frac{\partial L}{\partial x_{4n+j} \partial x_{5n+i}} dx_{4n+j} \wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} dx_{4n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} dx_{4n+j} \wedge dx_{5n+i} - \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} dx_{4n+j} \wedge dx_{6n+i}$$

$$+\frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} dx_{4n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{n+i}} dx_{5n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{5n+j} \partial x_i} dx_{5n+j} \wedge dx_{n+i}$$

$$-\frac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} dx_{5n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{5n+i}} dx_{5n+j} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} dx_{5n+j} \wedge dx_{4n+i}$$

$$+\frac{\partial^2 L}{\partial x_{5n+j} \partial x_{3n+i}} dx_{5n+j} \wedge dx_{5n+i} - \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{7n+i}} dx_{5n+j} \wedge dx_{6n+i} + \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{6n+i}} dx_{5n+j} \wedge dx_{7n+i}$$

$$-\frac{\partial^2 L}{\partial x_{6n+j} \partial x_{n+i}} dx_{6n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{6n+j} \partial x_i} dx_{6n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{4n+i}} dx_{6n+j} \wedge dx_{2n+i}$$

$$-\frac{\partial^2 L}{\partial x_{6n+j} \partial x_{5n+i}} dx_{6n+j} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{2n+i}} dx_{6n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{3n+i}} dx_{6n+j} \wedge dx_{5n+i}$$

$$-\frac{\partial^2 L}{\partial x_{6n+j} \partial x_{7n+i}} dx_{6n+j} \wedge dx_{6n+i} + \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{6n+i}} dx_{6n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{n+i}} dx_{7n+j} \wedge dx_i$$

$$+\frac{\partial^2 L}{\partial x_{7n+j} \partial x_i} dx_{7n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{4n+i}} dx_{7n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{5n+i}} dx_{7n+j} \wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{7n+j} \partial x_{2n+i}} dx_{7n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{3n+i}} dx_{7n+j} \wedge dx_{5n+i} - \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{7n+i}} dx_{7n+j} \wedge dx_{6n+i}$$

$$+\frac{\partial^2 L}{\partial x_{7n+j} \partial x_{6n+i}} dx_{7n+j} \wedge dx_{7n+i}$$

Let ξ be the second order differential equation by given **Eq.** (1) and defined by **Eq.** (6)

and

$$
\begin{aligned}
i_\xi\Phi_L^{J_1} = & -X^i\frac{\partial^2 L}{\partial x_j \partial x_{n+i}}\delta_i^j dx_i + X^i\frac{\partial^2 L}{\partial x_j \partial x_{n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_j \partial x_i}\delta_i^j dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_j \partial x_i}dx_j \\
& -X^i\frac{\partial^2 L}{\partial x_j \partial x_{4n+i}}\delta_i^j dx_{2n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_j \partial x_{4n+i}}dx_j - X^i\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}\delta_i^j dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}dx_j \\
& +X^i\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}}\delta_i^j dx_{4n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}\delta_i^j dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_j \\
& -X^i\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}\delta_i^j dx_{6n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}\delta_i^j dx_{7n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}dx_j \\
& -X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}}\delta_{n+i}^{n+j} dx_i + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}}dx_{n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_i}\delta_{n+i}^{n+j} dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_i}dx_{n+j} \\
& -X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}}\delta_{n+i}^{n+j} dx_{2n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}}dx_{n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}}\delta_{n+i}^{n+j} dx_{3n+i} \\
& +X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}}dx_{n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}}\delta_{n+i}^{n+j} dx_{4n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}}dx_{n+j} \\
& +X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}}\delta_{n+i}^{n+j} dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}}dx_{n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}}\delta_{n+i}^{n+j} dx_{6n+i} \\
& +X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}}dx_{n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}}\delta_{n+i}^{n+j} dx_{7n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}}dx_{n+j} \\
& - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}}\delta_{2n+i}^{2n+j} dx_i + X^i\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}}dx_{2n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_i}\delta_{2n+i}^{2n+j} dx_{n+i} \\
& -X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_i}dx_{2n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}}\delta_{2n+i}^{2n+j} dx_{2n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}}dx_{2n+j} \\
& -X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}}\delta_{2n+i}^{2n+j} dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}}dx_{2n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}}\delta_{2n+i}^{2n+j} dx_{4n+i} \\
& -X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}}dx_{2n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}}\delta_{2n+i}^{2n+j} dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}}dx_{2n+j} \\
& -X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}}\delta_{2n+i}^{2n+j} dx_{6n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}}dx_{2n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}}\delta_{2n+i}^{2n+j} dx_{7n+i} \\
& -X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}}dx_{2n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}\delta_{3n+i}^{3n+j} dx_i + X^i\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}dx_i \\
& +X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}\delta_{3n+i}^{3n+j} dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}dx_{3n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}}\delta_{3n+i}^{3n+j} dx_{2n+i} \\
& +X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}}dx_{3n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}}\delta_{3n+i}^{3n+j} dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}}dx_{3n+j} \\
& +X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}\delta_{3n+i}^{3n+j} dx_{4n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}dx_{3n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}\delta_{3n+i}^{3n+j} dx_{5n+i} \\
& -X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}dx_{3n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}}\delta_{3n+i}^{3n+j} dx_{6n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}}dx_{3n+j} \\
& +X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}}\delta_{3n+i}^{3n+j} dx_{7n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}}dx_{3n+j} - X^{4n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_{n+i}}\delta_{4n+i}^{4n+j} dx_i \\
& +X^i\frac{\partial L}{\partial x_{4n+j} \partial x_{n+i}}dx_{4n+j} + X^{4n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_i}\delta_{4n+i}^{4n+j} dx_{n+i} - X^{n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_i}dx_{4n+j} \\
& -X^{4n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_{4n+i}}\delta_{4n+i}^{4n+j} dx_{2n+i} + X^{2n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_{4n+i}}dx_{4n+j} - X^{4n+i}\frac{\partial L}{\partial x_{4n+j} \partial x_{5n+i}}\delta_{4n+i}^{4n+j} dx_{3n+i}
\end{aligned}
$$

$$+X^{3n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{4n+i}-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{5n+i}-X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j}-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{6n+i}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j}+\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_i+X^i\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}\delta_{5n+i}^{5n+j}dx_{n+i}$$

$$-X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{5n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i}+X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i}+X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{4n+i}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{5n+i}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{6n+i}+X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_i+X^i\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}\delta_{6n+i}^{6n+j}dx_{n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{6n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i}$$

$$+X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i}+X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{4n+i}-X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_i$$

$$+X^i\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}\delta_{7n+i}^{7n+j}dx_{n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{7n+j}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i}+X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i}$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{4n+i}\quad -X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{5n+i}-X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}$$

Since the closed standard Cliffordian Kähler form $\Phi_L^{J_1}$ on $(\mathbf{R}^8, V)$ is the symplectic structure, it is found

$$E_L^{J_1}=V_{J_1}(L)-L=X^i\frac{\partial L}{\partial x_{n+i}}-X^{n+i}\frac{\partial L}{\partial x_i}+X^{2n+i}\frac{\partial L}{\partial x_{4n+i}}+X^{3n+i}\frac{\partial L}{\partial x_{5n+i}}$$
$$-X^{4n+i}\frac{\partial L}{\partial x_{2n+i}}-X^{5n+i}\frac{\partial L}{\partial x_{3n+i}}+X^{6n+i}\frac{\partial L}{\partial x_{7n+i}}-X^{7n+i}\frac{\partial L}{\partial x_{6n+i}}-L$$

and hence

$$dE_L^{J_1} = X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j - X^{n+i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j + X^{2n+i}\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_j$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_j - X^{4n+i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j - X^{5n+i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_j - X^{7n+i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j}$$

$$-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{n+j}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{n+j}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{n+j} + X^i\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j}$$

$$+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} - X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{2n+j}$$

$$+X^i\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j}$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} - X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j} + X^i\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j}$$

$$-X^{n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}dx_{4n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} + X^i\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{5n+j}$$

$$+X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j} - X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}$$

$$+X^i\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{6n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j} - X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j} + X^i\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}$$

$$-X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{7n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j} - \frac{\partial L}{\partial x_j}dx_j - \frac{\partial L}{\partial x_{n+j}}dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}-\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}-\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}-\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}-\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}$$

With the use of **Eq.** (1), the following expressions can be obtained:

$$-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_{n+j}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_{2n+j}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_{3n+j}$$

$$+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_{4n+j}+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{5n+j}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_{6n+j}$$

$$+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_{7n+j}\quad-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_j+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}$$

$$-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{2n+j}\quad-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{3n+j}\quad+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{5n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{6n+j}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{7n+j}$$

$$-\;X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_j+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j}$$

$$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{3n+j}\quad+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{4n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{5n+j}$$

$$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{6n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{7n+j}\quad-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_j$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{2n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j}$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{4n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{5n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{6n+j}$$

$$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{7n+j}\quad-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}dx_j\quad+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_i}dx_{n+j}$$

$$-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{2n+j}\quad-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{3n+j}\quad+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{5n+j}\quad-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{6n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{7n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_j\quad+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{2n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{3n+j}\quad+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{4n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{6n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{7n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_j$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{2n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{3n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{4n+j}\quad+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{5n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{7n+j}\qquad-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_j+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{n+j}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{2n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{3n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{5n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{6n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}$$

$+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}$

$+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0$

If a curve denoted by $\alpha:\mathbf{R}\to\mathbf{R}^8$ is considered to be an integral curve of ξ, then we calculate the following equation:

$$
\begin{aligned}
&-X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_j-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_j\\
&-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_j-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}dx_j-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_j\\
&-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_j-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_j+X^i\frac{\partial^2 L}{\partial x_j\partial x_i}dx_{n+j}\\
&+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{n+j}\\
&+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_i}dx_{n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{n+j}\\
&+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{n+j}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_{2n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{2n+j}\\
&-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{2n+j}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{2n+j}\\
&-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{2n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{2n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{2n+j}\\
&-X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_{3n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{3n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{3n+j}\\
&-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{3n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{3n+j}\\
&-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{3n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{3n+j}+X^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_{4n+j}\\
&+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{4n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{4n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{4n+j}\\
&+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{4n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{4n+j}\\
&+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{4n+j}+X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{5n+j}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{5n+j}\\
&+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{5n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{5n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{5n+j}\\
&+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{5n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{5n+j}\\
&-X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_{6n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{6n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{6n+j}\\
&-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{6n+j}-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{6n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{6n+j}\\
&-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{6n+j}+X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_{7n+j}
\end{aligned}
$$

$$+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{7n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{7n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{7n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{7n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{7n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{7n+j}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}$$

$$+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0$$

alternatively

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}]dx_j+\frac{\partial L}{\partial x_j}dx_j$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_i}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_i}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}]dx_{n+j}+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}$$

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}]dx_{2n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{5n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}]dx_{3n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}]dx_{4n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}]dx_{5n+j}+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}$$

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}]dx_{6n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}]dx_{7n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0$$

Then we obtain the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{n+i}}=0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_i}=0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{4n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0, \tag{11}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_{7n+i}}=0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)-\frac{\partial L}{\partial x_{6n+i}}=0,$$

such that the equations obtained in **Eq.** (11) are said to be *Euler-Lagrange equations* structured on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$ by means of $\Phi_L^{J_1}$ and in the case, the triple $(\mathbf{R}^8, \Phi_L^{J_1}, \xi)$ is called a *mechanical system* on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$.

Secondly, we find Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^G on the standard Cliffordian Kähler manifold (M, V).

Consider J_2 be another local basis component on the Cliffordian Kähler manifold $(\mathbf{R}^8, V)$. Let ξ take as in **Eq.** (6). In the case, the vector field given by

$$\begin{aligned} V_{J_2} = J_2(\xi) = X^i\frac{\partial}{\partial x_{2n+i}} - X^{n+i}\frac{\partial}{\partial x_{4n+i}} - X^{2n+i}\frac{\partial}{\partial x_i} + X^{3n+i}\frac{\partial}{\partial x_{6n+i}} \\ +X^{4n+i}\frac{\partial}{\partial x_{n+i}} - X^{5n+i}\frac{\partial}{\partial x_{7n+i}} - X^{6n+i}\frac{\partial}{\partial x_{3n+i}} + X^{7n+i}\frac{\partial}{\partial x_{5n+i}}, \end{aligned} \tag{12}$$

is *Liouville vector field* on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$. The function given by $E_L^{J_2} = V_{J_2}(L) - L$ is *energy function.* Then the operator i_{J_2} induced by J_2 and denoted by

$$i_{J_2}\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., J_2X_i, ..., X_r) \tag{13}$$

is *vertical derivation,* where $\omega \in \wedge^r\mathbf{R}^8$, $X_i \in \chi(\mathbf{R}^8)$. The *vertical differentiation* d_{J_2} are defined by

$$d_{J_2} = [i_{J_2}, d] = i_{J_2}d - di_{J_2}. \tag{14}$$

Since taking into considering J_2, the closed standard Clifford Kähler form is the closed 2-form

given by $\Phi_L^{J_2} = -dd_{J_2}L$ such that

$$\begin{aligned} d_{J_2} &= \frac{\partial}{\partial x_{2n+i}}dx_i - \frac{\partial}{\partial x_{4n+i}}dx_{n+i} - \frac{\partial}{\partial x_i}dx_{2n+i} + \frac{\partial}{\partial x_{6n+i}}dx_{3n+i} \\ &+ \frac{\partial}{\partial x_{n+i}}dx_{4n+i} - \frac{\partial}{\partial x_{7n+i}}dx_{5n+i} - \frac{\partial}{\partial x_{3n+i}}dx_{6n+i} + \frac{\partial}{\partial x_{5n+i}}dx_{7n+i} \end{aligned} \tag{15}$$

and defined by operator

$$d_{J_2} : \mathcal{F}(\mathbf{R}^8) \rightarrow \wedge^1\mathbf{R}^8. \tag{16}$$

The closed standard Clifford Kähler form $\Phi_L^{J_2}$ on $\mathbf{R}^8$ is the symplectic structure. So it holds

$$\begin{aligned} E_L^{J_2} &= V_{J_2}(L) - L = X^i\frac{\partial L}{\partial x_{2n+i}} - X^{n+i}\frac{\partial L}{\partial x_{4n+i}} - X^{2n+i}\frac{\partial L}{\partial x_i} + X^{3n+i}\frac{\partial L}{\partial x_{6n+i}} \\ &+ X^{4n\ +i}\frac{\partial L}{\partial x_{n+i}} - X^{5n+i}\frac{\partial L}{\partial x_{7n+i}} - X^{6n+i}\frac{\partial L}{\partial x_{3n+i}} + X^{7n+i}\frac{\partial L}{\partial x_{5n+i}} - L \end{aligned} \tag{17}$$

By means of **Eq.** (1), using (6), (15) and (17), also taking into consideration the above first part we calculate the equations

$$\begin{gathered} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{4n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_i} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) + \frac{\partial L}{\partial x_{6n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right) + \frac{\partial L}{\partial x_{n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right) - \frac{\partial L}{\partial x_{7n+i}} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right) - \frac{\partial L}{\partial x_{3n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right) + \frac{\partial L}{\partial x_{5n+i}} = 0, \end{gathered} \tag{18}$$

Hence the equations obtained in **Eq.** (18) are called *Euler-Lagrange equations* structured by means of $\Phi_L^{J_2}$ on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$ and so, the triple $(\mathbf{R}^8, \Phi_L^{J_2}, \xi)$ is said to be a *mechanical system* on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$.

Thirdly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_3}$ on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$.

Let J_3 be a local basis on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$. Let semispray ξ give as in **Eq.**(6). Therefore, *Liouville vector field* on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$ is the vector field given by

$$V_{J_3}=J_3(\xi)=X^i\frac{\partial}{\partial x_{3n+i}}-X^{n+i}\frac{\partial}{\partial x_{5n+i}}-X^{2n+i}\frac{\partial}{\partial x_{6n+i}}-X^{3n+i}\frac{\partial}{\partial x_i}$$
$$+X^{4n+i}\frac{\partial}{\partial x_{7n+i}}+X^{5n+i}\frac{\partial}{\partial x_{n+i}}+X^{6n+i}\frac{\partial}{\partial x_{2n+i}}-X^{7n+i}\frac{\partial}{\partial x_{4n+i}}. \tag{19}$$

The function given by $E_L^{J_3}=V_{J_3}(L)-L$ is *energy function* and calculated by

$$E_L^{J_3}=X^i\frac{\partial L}{\partial x_{3n+i}}-X^{n+i}\frac{\partial L}{\partial x_{5n+i}}-X^{2n+i}\frac{\partial L}{\partial x_{6n+i}}-X^{3n+i}\frac{\partial L}{\partial x_i}$$
$$+X^{4n+i}\frac{\partial L}{\partial x_{7n+i}}+X^{5n+i}\frac{\partial L}{\partial x_{n+i}}+X^{6n+i}\frac{\partial L}{\partial x_{2n+i}}-X^{7n+i}\frac{\partial L}{\partial x_{4n+i}}-L. \tag{20}$$

The function i_{J_3} induced by J_3 and shown by

$$i_{J_3}\omega(X_1,X_2,...,X_r)=\sum_{i=1}^{r}\omega(X_1,...,J_3X_i,...,X_r), \tag{21}$$

is said to be *vertical derivation,* where $\omega\in\wedge^r\mathbf{R}^8$, $X_i\in\chi(\mathbf{R}^8)$. The *vertical differentiation* d_{J_3} is denoted by

$$d_{J_3}=[i_{J_3},d]=i_{J_3}d-di_{J_3}, \tag{22}$$

Considering J_3 , the closed Kähler form is the closed 2-form given by $\Phi_L^{J_3}=-dd_{J_3}L$ such that

$$\begin{aligned} d_{J_3} \;=\; & \frac{\partial}{\partial x_{3n+i}}dx_i-\frac{\partial}{\partial x_{5n+i}}dx_{n+i}-\frac{\partial}{\partial x_{6n+i}}dx_{2n+i}-\frac{\partial}{\partial x_i}dx_{3n+i}\\ &+\frac{\partial}{\partial x_{7n+i}}dx_{4n+i}+\frac{\partial}{\partial x_{n+i}}dx_{5n+i}+\frac{\partial}{\partial x_{2n+i}}dx_{6n+i}-\frac{\partial}{\partial x_{4n+i}}dx_{7n+i}\end{aligned}$$

and

$$d_{J_3}:\mathcal{F}(\mathbf{R}^8)\to\wedge^1\mathbf{R}^8 \tag{23}$$

Using **Eq.** (1), similar to the above first and second cases , we find the following expression the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{3n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{5n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{6n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{7n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0, \tag{24}$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0,$$

Thus the equations given in **Eq.** (24) infer *Euler-Lagrange equations* structured by means of $\Phi_L^{J_3}$ on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$ and therefore the triple $(\mathbf{R}^8, \Phi_L^{J_3}, \xi)$ is named a *mechanical system* on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$.

4 Conclusion

From above, Lagrangian mechanics has intrinsically been described taking into account a canonical local basis $\{J_1, J_2, J_3\}$ of V on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$.

The paths of semispray ξ on the standard Cliffordian Kähler manifold are the solutions Euler–Lagrange equations raised in (11), (18) and (24), and also obtained by a canonical local basis $\{J_1, J_2, J_3\}$ of vector bundle V on the standard Cliffordian Kähler manifold $(\mathbf{R}^8, V)$. One can be proved that these equations are very important to explain the rotational spatial mechanics problems.

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